Toda Lattice Models with Boundary

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Abstract

We consider the soliton solutions in 1- and (1+1)-dimensional Toda lattice models with a boundary. We make use of the solutions already known on a full line by means of the Hirota's method. We explicitly construct the solutions satisfying the boundary conditions. The $\mathbf{Z}_\infty$-symmetric boundary condition can be introduced by the two-soliton solutions naturally.

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1 Introduction

Integrable models on a half line have been widely studied recently. To investigate the integrable models on a half line, two approaches are already known. One is the algebraic approach by the boundary Yang-Baxter formalism[1] and the other is the field theoretical one with the Lagrangian accompanied by the proper boundary terms[2]. The model which is integrable on a full line needs the proper boundary terms in order to be integrable also on a half line. Therefore, the boundary terms should be determined by the requirement that the infinite number of integrals of motion on a full line must survive after introducing the boundary. Both approaches assume that the particle content and the S-matrix are the same as those on a full line. Though they offer quite beautiful descriptions of the integrable models on a half line, especially in sine-Gordon model [3, 4] and affine Toda field theory (ATFT) [5], the relations between the two approaches are to be clarified. As for the sine-Gordon model, the semi-classical analysis has already been applied to determine the relation of the boundary term and the boundary S-matrix [6].

In the Lagrangian approach, there are two ways to demand the infinite number of integrals of motion to survive after introducing the boundary. In the first, the given boundary terms determine the boundary condition of the variables or fields through the Euler-Lagrange equation. These boundary terms give the boundary condition of the conserved currents, which have the same forms as those in a full line. Therefore it is possible to define the integrals of the conserved currents over the half line, which do not have the asymptotic condition. If we demand those integrals to be time-independent, we can determine the boundary terms inversely. The second way is to use the transfer matrix. We make the proper image of the half line at the boundary by adding the proper boundary terms to the Lagrangian, which enable us to consider the trace over the spatial direction. Therefore we can define the transfer matrix on a half line properly and demand it to be time-independent.

On the other hand, Hirota's method has been used to obtain the soliton solutions of the equation of motion in a variety of integrable models on a full line in 1- or (1+1)-dimensions, for example, the nonlinear Schrödinger model, sine-Gordon model, KdV model and Toda lattice model[7]. This method gives the solutions in quite an explicit form so it is an appropriate method to consider the boundary conditions.

In this paper, we discuss the classical solutions of the equation of motion with the integrable boundary conditions in 1-dimensional Toda lattice model and the extension to the (1+1)-dimensional field theory (Toda lattice field theory). As for the other integrable models on a half line, the classical solutions are already known by the *classical inverse scattering method*[9, 8]. Therefore our analysis of Toda lattice models is considered as a first step.

This paper is organized as follows. In section 2 the Toda lattice (TL) model is investigated. The TL model is considered as the multi-particle systems or the model of an electric circuit with solitons. The equation of motion has already been solved by means of Hirota's method [7]. On the other hand, the boundary terms to make the TL model integrable on a half line have been determined by means of the transfer matrix method[2]. We will consider the soliton solutions obtained by Hirota's method on a full line and satisfying the boundary condition explicitly. We will see the meaning of the parameter in the boundary term in several simple

examples. In section 3 the (1+1)-dimensional Toda lattice field theory (TLFT) is introduced to clarify the notations. The TLFT is an extension of the TL model to a field theory in (1+1)-dimension and has the soliton solutions unlike the $a_N^{(1)}$ ATFT with a real coupling constant. In section 4 the TLFT with a proper boundary term on a half line is investigated. There seem to be several ways to introduce the boundary in this model, for example, the half number of the fields should vanish. However, we consider the way to introduce the boundary in the spatial direction, which has the connection with the $a_N^{(1)}$ ATFT analyzed already, therefore it is the most suitable to consider. We will see that the boundary terms conserving the $\mathbf{Z}_\infty$-symmetry can be induced by the two-soliton solutions. In section 5 we present a summary of this paper.

2 Integrable Boundary Condition in Toda Lattice Model

It is well-known that the one-dimensional TL model is integrable, which can be shown for example in terms of the construction by the Lax pair. If we consider the integrable models on a half line, integrability imposes the restricting condition on the boundary terms, which determines the boundary condition of the model. The integrable boundary condition (IBC), which is compatible with the integrability, can be determined by the requirement that the integral of motion on a full line must survive also on a half line. In turn, this can be realized for example by imposing that the transfer matrix, which is validly defined on a half line, to be time-independent like that without the boundary. The IBC in TL model is determined in Sklyanin's paper[2] in this scheme. In this section, we consider the classical solutions with the IBC according to Hirota's method.

To begin with, we consider the TL model on a full line. The one-dimensional TL model is defined with the canonical coordinates (π_n, q_n), which are functions of the time variable t and on a full line, integer $n \in (-\infty, \infty)$. The hamiltonian and the Poisson brackets are

$$H = \sum_{n=-\infty}^{\infty} \frac{\pi_n^2}{2} + \sum_{n=-\infty}^{\infty} e^{q_{n+1}-q_n}, \qquad \{\pi_m, q_n\} = \delta_{nm}. \tag{1}$$

Now we write down the equation of motion in terms of new variables $r_n = q_{n+1} - q_n$. The equation of motion is

$$\ddot{r}_n = e^{r_{n+1}} - 2e^{r_n} + e^{r_{n-1}}, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{2}$$

here $\dot{}$ represents the time derivative $\frac{d}{dt}$. We can obtain the soliton solutions of the equation of motion (2) by means of Hirota's method. To apply Hirota's method, we introduce the new variables V_n and F_n as $r_n = \ln(1 + V_n)$ and $V_n = \frac{d^2}{dt^2} \ln F_n$. With these variables V_n and F_n, the equation of motion (2) can be written in a simpler form

$$\ddot{F}_n F_n - \dot{F}_n^2 = F_{n+1} F_{n-1} - F_n^2. \tag{3}$$

The above equation of motion (3) has s-soliton solutions

$$\begin{aligned} F_n &= \sum_{\underline{\mu}=0,1} \exp\left(\sum_{i=1}^{s} \mu_i \eta_i + \sum_{i<j} A_{ij} \mu_i \mu_j \right), \\ \eta_i &= p_i n - \Omega_i t + \delta_i, \end{aligned} \tag{4}$$

where δ_i's are arbitrary real parameters and μ_i $(i = 1, 2, \ldots, s)$ is 0 or 1. The real-valued momenta (p_i, Ω_i) and the coefficient A_{ij} must satisfy the condition

$$\begin{aligned}
\Omega_i &= 2\epsilon_i \sinh \frac{p_i}{2}, \qquad \epsilon_i = \pm 1, \\
e^{A_{ij}} &= \begin{cases} \left(\dfrac{\sinh \frac{1}{4}(p_i - p_j)}{\sinh \frac{1}{4}(p_i + p_j)}\right)^2 & \epsilon_i \epsilon_j > 0 \\ \left(\dfrac{\cosh \frac{1}{4}(p_i - p_j)}{\cosh \frac{1}{4}(p_i + p_j)}\right)^2 & \epsilon_i \epsilon_j < 0 \end{cases}.
\end{aligned} \tag{5}$$

The above soliton solutions approach zero as $|n| \to \infty$ and does not diverge anywhere provided p_i, c_i and δ_i are real numbers [11, 12].

We are ready to consider the TL model with the IBC. The hamiltonian takes the form

$$H = \sum_{n=1}^{\infty} \frac{\pi_n^2}{2} + \sum_{n=1}^{\infty} e^{q_{n+1} - q_n} + \left(\alpha e^{q_1} + \frac{\beta}{2} e^{2q_1}\right). \tag{6}$$

It can be shown that the model given by the above hamiltonian is also integrable with the arbitrary parameters α and β because it has an infinite number of the integrals of motion[2]. The equation of motion can be written down with $r_n = q_{n+1} - q_n$ $(n = 1, 2, \ldots)$ and q_1 as

$$\ddot{r}_n = e^{r_{n+1}} - 2e^{r_n} + e^{r_{n-1}} \qquad n = 2, 3, \cdots, \tag{7}$$
$$\ddot{r}_1 = e^{r_2} - 2e^{r_1} + (\alpha e^{q_1} + \beta e^{2q_1}). \tag{8}$$

Eqs.(7) for r_n $(n = 2, 3, \ldots)$ are the same as those on a full line (2) and Eq.(8) for r_1 includes the variable q_1. We make a remark that if $\lim_{n\to\infty} |\sum_{i=1}^{n} r_i| < \infty$, which is true for the soliton solution (4), $q_1 = (q_1 - q_2) + (q_2 - q_3) + \ldots = -\sum_{i=1}^{\infty} r_i$. Comparing Eqs.(2) and (7)(8), we obtain the boundary condition in Eq.(2), which enables the soliton solution (4) on a full line to satisfy Eqs.(7) and (8) on a half line in the form

$$e^{r_0} = \alpha \prod_{i=1}^{\infty} e^{-r_i} + \beta \left(\prod_{i=1}^{\infty} e^{-r_i}\right)^2. \tag{9}$$

In terms of F_n, this can be written down simply as

$$F_{-1} = \alpha F_0 + \beta F_1. \tag{10}$$

Firstly, we will concentrate the special case with $\alpha + \beta = 1, \beta \geq 0$ for simplicity. In this case, we can find two-soliton solutions with arbitrary momentum satisfying the boundary condition (10). We consider the two-soliton solution

$$\begin{aligned}
F_n &= 1 + e^{-\Omega t}(e^{pn+\delta_1} + e^{-pn+\delta_2}) + \left(\cosh \frac{p}{2}\right)^2 e^{-2\Omega t + \delta_1 + \delta_2}, \\
\Omega &= 2 \sinh \frac{p}{2},
\end{aligned} \tag{11}$$

with an arbitrary momentum p. The boundary condition(10) gives the equation of the initial displacements δ_1 and δ_2

$$e^{\delta_1-\delta_2} = \frac{1+\beta e^{-p}}{e^{-p}+\beta}, \tag{12}$$

which is positive-definite even if $\alpha = 1-\beta$ is negative. Therefore the system is stable[10]. This equation means the following. We assume $\Omega > 0, p > 0$. In the limit $t \to -\infty$, there exist two solitons with momenta (Ω, p) and $(\Omega, -p)$. As the time t grows to ∞, F_n goes to 1. Therefore if the initial displacements of the two solitons, δ_1 and δ_2, satisfy Eq.(12), these two solitons undergo a pair-annihilation due to the boundary. In particular, if $\alpha = 0$ and $\beta = 1$, Eq.(12) is simplified as $\delta_1 = \delta_2$. It means that in this case, the shift of the initial displacements due to the boundary is independent of the momentum p of the soliton.

Before closing this section, we briefly mention the general case with arbitrary α and β. If $\alpha + \beta \neq 1$, it is much more difficult to find the solutions satisfying the boundary condition (10) than those with $\alpha + \beta = 1$. We consider the case in which the initial momentum p takes a special value so that the two-soliton solutions may exist. We consider a two-soliton solution

$$\begin{aligned} F_n &= 1 + e^{\Omega t - pn + \delta_1} + e^{-\Omega t - pn + \delta_2} + \left(\cosh\frac{p}{2}\right)^{-2} e^{-2pn+\delta_1+\delta_2}, \\ \Omega &= 2\sinh\frac{p}{2}. \end{aligned} \tag{13}$$

If we assume the region of β as $-e^{-p} < \beta < -e^{-2p}$ for $p > 0$ or $-e^{-2p} < \beta < -e^{-p}$ for $p < 0$, this soliton-solution satisfies the boundary condition (10) only if

$$e^{-p} - \alpha + |\beta| e^{p} = 0, \tag{14}$$

$$e^{\delta_1+\delta_2} = \frac{|\beta| - e^{-p}}{-|\beta| e^{p} + e^{-2p}}. \tag{15}$$

where the value (15) is made positive. We assume $\Omega > 0, p > 0$ as before. Considering the limits $t \to \pm\infty$, the initial soliton with momentum (Ω, p) is scattered by the boundary to the soliton with momentum $(-\Omega, p)$ if the momentum p satisfies Eq.(14) and the initial phase δ_2 are determined from δ_1 by Eq.(15).

3 Definitions and Solutions in (1+1)-dimensional Toda Lattice Field Theory

In this section, we consider the extension of one-dimensional TL model to a (1+1)- dimensional field theory (TLFT). As well as other (1+1)-dimensional integrable classical field theories, the equation of motion of the TLFT can also be solved by means of Hirota's method. (1+1)-dimensional TLFT ($a^{(1)}_{\infty}$ Affine Toda field theory with the real coupling constant) is defined by the action

$$\mathcal{S} = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau \mathcal{L} = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2}(\partial_\tau \vec{\phi})^2 - \frac{1}{2}(\partial_\sigma \vec{\phi})^2 - \sum_{i=-\infty}^{\infty} e^{\vec{\alpha}_i \cdot \vec{\phi}} \right). \tag{16}$$

Here we explain the notations briefly. Let $\mathbf{R}^\infty$ be the infinite-dimensional vector space. $\vec{\phi}(\sigma,\tau) \in \mathbf{R}^\infty$ is an infinite-dimensional vector field and $\vec{\alpha}_i = \vec{e}_{i+1} - \vec{e}_i$, where $\{\vec{e}_i\}$ is the standard basis of $\mathbf{R}^\infty$, is a simple root of $a_\infty^{(1)}$ Lie algebra. If we put $\vec{\alpha}_i \cdot \vec{\phi} = \phi_i$, the equation of motion can be written as

$$\partial_\tau^2 \phi_n - \partial_\sigma^2 \phi_n = 2e^{\phi_n} - e^{\phi_{n+1}} - e^{\phi_{n-1}}, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{17}$$

We carry out the calculation in the light-cone coordinate $z = (\sigma - \tau)/2$ and $\bar{z} = (\sigma + \tau)/2$ $(\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}})$ to keep the notation simple. If we introduce new variables F_n and V_n $\phi_n = \ln(1 + V_n)$ and $V_n = -\partial\bar{\partial} \ln F_n$ as before, the equation of motion of the model (16) can be written in terms of F_n's as

$$\partial F_n \bar{\partial} F_n - F_n \partial\bar{\partial} F_n = F_{n+1} F_{n-1} - F_n^2. \tag{18}$$

The equation of motion (18) has the s-soliton solutions

$$\begin{aligned} F_n &= \sum_{\underline{\mu}=0,1} \exp\left(\sum_{i=1}^{s} \mu_i \rho_i + \sum_{i<j} A_{ij} \mu_i \mu_j \right), \\ \rho_i &= a_i z + b_i \bar{z} + c_i n + \delta_i, \end{aligned} \tag{19}$$

here the initial phases δ_i are arbitrary and the momenta a_i,b_i and c_i and the coefficients A_{ij} must satisfy the condition

$$\begin{aligned} a_i b_i &= -4 \sinh^2 \frac{c_i}{2}, \\ e^{A_{ij}} &= -\frac{(a_i - a_j)(b_i - b_j) + 4 \sinh^2 \dfrac{c_i - c_j}{2}}{(a_i + a_j)(b_i + b_j) + 4 \sinh^2 \dfrac{c_i + c_j}{2}}. \end{aligned} \tag{20}$$

Several remarks are in order. Firstly, from the requirement that the solutions given by (18) do not diverge on a full line, *i.e.* $V_n > -1$, we will get the condition that $e^{A_{ij}} \geq 0$ and that δ_i's are real values. Secondly, the above given solutions are soliton solutions. It is well-known that contrary to the sine-Gordon case, there is no soliton solutions in the (real-coupling) $a_N^{(1)}$ ATFT for finite N, for example sinh-Gordon model. The reason is that in the case of $a_N^{(1)}$ ATFT for the finite N, the cyclic condition in the direction of the affine coordinate, that is $i + N \equiv i$, demands the momenta c_i to take certain complex values $c_i = (2\pi i/N)\mathbf{Z}$. This fact causes the naive soliton solutions given in Eq.(19) to diverge to infinity in the limit $\sigma \to \pm\infty$, therefore these models do not have soliton solutions. The TLFT, in spite of corresponding to the $a_\infty^{(1)}$ ATFT with the real coupling constant, has no cyclic condition in the affine direction so we can choose c_i to be real values, which do not cause the divergence. Conversely, we cannot obtain the solutions in TLFT by merely taking the limit of $N \to \infty$ in $a_N^{(1)}$ ATFT. We need to take the analytic continuation in the affine direction.

4 Toda Lattice Field Theory with Boundary

In this section, we will consider the TLFT on a half line. We choose the space coordinate σ in $(-\infty, 0)$ and the time coordinate τ in $(-\infty, \infty)$. The action of the TLFT on a half line is given by

$$\mathcal{S} = \int_{-\infty}^{0} d\sigma \int_{-\infty}^{\infty} d\tau \mathcal{L} + \int_{-\infty}^{\infty} d\tau \mathcal{L}_B, \tag{21}$$

here the *bulk* Lagrangian density $\mathcal{L}$ is the same as that given before on a full line (16) and the boundary term $\mathcal{L}_B$ is given as

$$\mathcal{L}_B = -\sum_{i=-\infty}^{\infty} \mathcal{A}_i e^{\vec{\alpha}_i \cdot \vec{\phi}/2}|_{\sigma=0}, \tag{22}$$

with the parameters $\mathcal{A}_i$. It is well-known that the $a_N^{(1)}$-ATFT with a finite $N \geq 2$ defined by the above action (22) is integrable if $|\mathcal{A}_i| = 0$ or 2 for all i[5]. In these cases, it is shown that the integrals of motion on a full line can survive after introducing the boundary at $\sigma = 0$. Now we consider the classical (soliton) solutions in this model on a half line. The equation of motion is the same as that on a full line (17) and the boundary term $\mathcal{L}_{\mathcal{B}}$ in the action (21) induces the boundary condition of $\vec{\phi}$ at $\sigma = 0$

$$\partial_\sigma \phi_n \Big|_{\sigma=0} = 2\mathcal{A}_n e^{\phi_n/2} - \mathcal{A}_{n+1} e^{\phi_{n+1}/2} - \mathcal{A}_{n-1} e^{\phi_{n-1}/2} . \tag{23}$$

For the later convenience, we write down the boundary condition (23) in terms of the variable F_n, which is defined the same as that on a full line,

$$\left(2\frac{F_n'}{F_n} - \frac{F_{n+1}'}{F_{n+1}} - \frac{F_{n-1}'}{F_{n-1}}\right)|_{\sigma=0} = \left(\mathcal{A}_n \frac{\sqrt{F_{n+1}F_{n-1}}}{F_n} - \frac{\mathcal{A}_{n+1}}{2}\frac{\sqrt{F_{n+2}F_n}}{F_{n+1}} - \frac{\mathcal{A}_{n-1}}{2}\frac{\sqrt{F_{n-2}F_n}}{F_{n-1}}\right)|_{\sigma=0}, \tag{24}$$

here $'$ means the spatial derivative $\partial/\partial\sigma$. We will consider the soliton solutions (19) obtained on a full line, some of which satisfy the boundary condition (24).

Firstly, we consider the simplest example; $\mathcal{A}_i = 0$ for all i. If a solution F_n is an even function in σ, it satisfies the boundary condition (24). In particular, if we consider two-soliton solutions ($s = 2$) in (19) with

$$\rho_1 = p\sigma - \epsilon\tau + cn + \delta, \qquad \rho_2 = -p\sigma - \epsilon\tau + cn + \delta, \tag{25}$$

the coefficient A_{12} takes quite a simple form as

$$e^{A_{12}} = \left(1 - \left(\frac{2}{p}\sinh^2\frac{c}{2}\right)^2\right)^{-1}, \tag{26}$$

from the condition (20). Providing this solution does not diverge on a half line, the coefficient (26) must be positive, that is $|p| \geq 2\sinh^2\frac{c}{2}$.

Secondly, we consider the general boundary condition given by (23). We assume two-soliton solutions ($s = 2$) given by (19). Because the lefthand-side of Eq.(24) is a rational

function of e^{ρ_1} and e^{ρ_2}, we must impose that the square root $\sqrt{F_{n+1}F_{n-1}}|_{\sigma=0}$ takes the form of a certain polynomial in e^{ρ_1} and e^{ρ_2}. This condition makes the calculation much simpler and gives the constraints of the momenta as

$$\begin{aligned}&\rho_1 = p\sigma - \epsilon\tau + cn + \delta_1, \qquad \rho_2 = p\sigma + \epsilon\tau - cn + \delta_2,\\&e^{A_{12}} = 1 - \left(\frac{2}{p}\sinh^2\frac{c}{2}\right)^2,\end{aligned} \tag{27}$$

where momenta p, ϵ and c are arbitrary real values with the condition (20) and the phase displacements δ_1 and δ_2 must satisfy the equation

$$e^{\delta_1+\delta_2} = \frac{1}{1+\gamma\left(\frac{2}{p}\sinh^2\frac{c}{2}\right)}, \qquad \gamma = \pm 1. \tag{28}$$

Due to the condition of the absence of the divergence, *i.e.* $V_n > -1$, we must insist $|p| \geq 2\sinh^2\frac{c}{2}$ as before. With the above preparation, we can consider the boundary condition (24). Because we can easily obtain

$$\left(F_n' + \gamma\sqrt{F_{n+1}F_{n-1}} - (p-\gamma)F_n\right)|_{\sigma=0} = 0, \tag{29}$$

we see that $\mathcal{A}_i = 2\gamma$ for all i. Therefore, if we consider the two-soliton solutions, we automatically obtain the $\mathbf{Z}_\infty$-preserving boundary condition, $\mathcal{A}_i = 2$ or -2 for all i.

We note that if only the soliton solutions are considered, it would be quite difficult to find the solutions satisfying the $\mathbf{Z}_\infty$-symmetry breaking boundary condition, for example $\mathcal{A}_0 = -2$ and $\mathcal{A}_i = 2$ for $i \neq 0$. In particular, even if we introduce the three-soliton solutions with

$$\begin{aligned}&\rho_1 = p\sigma - \epsilon\tau + cn + \delta_1, \quad \rho_2 = p\sigma + \epsilon\tau - cn + \delta_2,\\&\rho_3 = \pm 2\sigma + i\pi n + \delta_3,\end{aligned} \tag{30}$$

which is the unique choice for the three-soliton solution to satisfy the boundary condition(23) and to have no divergence, we obtain the same result as that in the case of the two-soliton solutions (29). It means that the third soliton given by (30) has no effect on the boundary condition.

5 Summary

In this paper, we have considered the TL model and TLFT with boundary and seen the meaning of the parameters in the boundary term. A few comments are in order. Firstly, the analysis in this paper can be applied only to the other models with the soliton solutions. Therefore, it is impossible to consider the $a_N^{(1)}$ ATFT with finite N and real coupling constant, which do not have the soliton solutions in the same way. However, it is possible to consider the $a_N^{(1)}$ ATFT with imaginary coupling constant. Especially, the analysis of TLFT with the Neumann-type boundary condition can be similarly used in that for $a_N^{(1)}$ ATFT with imaginary

coupling constant. Secondly, we did not analyze TL with arbitrary α and β nor TLFT with general $\mathcal{A}_i$'s. To analyze these models, we must take multi-soliton solutions into account.

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